Planar Dynamics of Free Rotating Flexible Beams with Tip Masses

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Analytical and numerical results are obtained for a free elastic rotating beam with tip masses which have rotatory inertia. The coordinate system which is employed has the advantage of rotating at a constant angular velocity, thereby simplifying the equations of motion. Orthogonality conditions are developed for the rotating system, and by employing these conditions it is shown that the two rigid-body modes have no contribution to the eigenfunction expansion for the deflection w, when w is expressed relative to the selected coordinates. Solutions for the rotating flexible modes and natural frequencies of an H-type configuration are developed by expressing each rotating mode as a series expansion of the nonrotating modes, employing the Rayleigh-Ritz method to obtain the modal coefficients. Numerical results illustrate an increase in each of the first five natural frequencies with angular velocity for a particular configuration. The influence of rotation on the first five mode shapes is also illustrated.

Nomenclature

a_1,a_2 = see Fig. 3
a_{cx},a_{cy} = x and y components of inertial acceleration of system c.m.
B_{ix},B_{iy} = external x and y body-force components on end compartment i
c.m. = center of mass
E = modulus of elasticity
f_x = externally applied force per unit length in x direction
$G(v)$ = linear operator defined by Eq. (30); $\tilde{G}(u_n) \equiv l^3G(v_n)/EI$
H = x component of internal force on beam cross section
h = magnitude of system angular momentum about system c.m.
I = area moment of inertia of beam cross section
i,j,k = unit vectors along x, y, z axes, respectively
J = mass moment of inertia of undeformed system about z axis
$J_{\zeta i}$ = moment of inertia of end compartment i about ζ_i axis
L_i = $L_i = l_i + a_i$, distance from system c.m. to c.m. of compartment i
l = $l = l_1 = l_2$ for H-type space station
l_i = length from system c.m. to point of contact of beam
M = internal bending moment on beam cross section
m = $m = m_1 = m_2$ for case of H-type space station
m_i = mass of end compartment $i(i = 1,2)$
N = number of terms in Rayleigh-Ritz expansion for u_n
$N(v_n)$ = linear operator defined by Eq. (31); $\tilde{N}(u_n) \equiv N(v_n)/\rho l$
N_n = normalization constant for eigenfunctions [see Eq. (38a)]
$\tilde{N}_n$ = $N_n/\rho l^3$, normalization constant for nondimensional eigenfunctions
P = see Eq. (7)
p,p_j = oscillation frequency for free system; p_j is jth frequency corresponding to mode $v_j(x)$
Q = see Eq. (10)
q = externally applied force per unit length in y direction
r = $m/\rho l$, tip to beam-mass ratio for H-type space station
r = vector from system c.m. to c.m. of elementary beam section of length dx
$\mathbf{r}_i$ = vector from system c.m. to c.m. of end compartment $i(i = 1,2)$
s = x/l
T_i = externally applied torque on end compartment
T_c = resultant torque on entire system about its c.m.
t = time
u_n,u_{no} = v_n/l, $u_{no} \equiv u_n$ for nonrotating system
V = y component of internal force on beam cross section
v,v_j = spatial coordinate in separation of variables solution; $v_j(x)$ is jth normal mode (eigenfunction corresponding to eigenvalue p_j)
w = deflection of neutral axis
w_i = deflection of c.m. of end compartment i $(i = 1,2)$
X = $X[y,z]$ is a scalar product defined by Eq. (57); $X[y] \equiv X[y,y]$
x,y,z = rotating coordinates with origin at c.m. of system
Z = $Z[y,z]$ is a scalar product defined by Eq. (58); $Z[y] \equiv Z[y,y]$
z_{ij} = $Z[u_{io},u_{jo}]$
β_n = $\lambda_{no}{}^{1/2}$
Γ = mass moment of inertia of beam per unit length
δ_{nj} = Kronecker δ, $\delta_{nj} = 0$ for $n = j$ and $\delta_{nn} = 1$
ϵ = $(\rho l^4/EI)\omega^2$
θ = angular position of y axis in space
λ_n,λ_{no} = $(\rho l^4/EI)p_n{}^2$, $\lambda_{no} \equiv \lambda_n$ for nonrotating system
μ = nondimensional rotatory inertia coefficient (J_ζ/ml^2)
ξ_i,η_i,ζ_i = c.m.-principal axes in compartment i $(i = 1,2)$
ρ = mass density of beam per unit length
τ = externally applied torque per unit length on beam
ω = mean angular velocity of x,y axes
$\langle v_j(x),v_n(x)\rangle$ = scalar product defined by Eq. (39)

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Introduction

Rotating vehicles are of current interest for manned space excursion because of the artificial gravitational environment provided by the centrifugal force field. As a result of rocket payload weight limitations and the large size required, these vehicles may be quite flexible.

Previous investigations of the motion of rotating flexible vehicles include a study of the transient behavior of various configurations which was obtained by lumped-mass methods together with Runge-Kutta numerical-integration techniques.[1,2] Also, in Ref. 3 exact nonlinear solutions were obtained for the axial oscillations of free double mass systems. Analytical solutions for the transverse oscillations of cable-connected systems were obtained by Chobotov[4] and Targoff.[5] In a recent survey of the problems involved,[6] Ashley briefly considers a rotating Euler beam with no tip masses as well as a variety of other large flexible bodies in orbit.

This paper describes an analytical investigation into the planar dynamics of a free rotating beam-connected system. This system is an idealization of a satellite which is composed of two rigid end compartments connected by a flexible beam with uniform mass distribution and elastic properties.

Formulation and Coordinates‡

Figure 1 illustrates the general system under consideration. The x,y axes are a moving set of axes to which the small oscillations of the system will be referred. The mass of a cross-sectional element of length dx is denoted by dm and the mass of the end compartments by m_1 and m_2. Since the origin of the x,y axes is selected so as to coincide with the system center of mass (c.m.),

$$\int_{-l_2}^{l_1} \mathbf{r}dm + m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0 \tag{1}$$

By geometry,

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}w \tag{2}$$

$$\mathbf{r}_1 = \mathbf{i}L_1 + \mathbf{j}w_1,\ w_1 \equiv w(l_1,t) + a_1 \frac{\partial w(l_1,t)}{\partial x} \tag{3}$$

$$\mathbf{r}_2 = -\mathbf{i}L_2 + \mathbf{j}w_2,\ w_2 \equiv w(-l_2,t) - a_2 \frac{\partial w(-l_2,t)}{\partial x} \tag{4}$$

With $dm = \rho dx$, the following two scalar relations are obtained from Eq. (1):

$$\tfrac{1}{2}\rho l_1^2 + m_1L_1 = \tfrac{1}{2}\rho l_2^2 + m_2L_2 \tag{5}$$

$$P[w] = 0 \tag{6}$$

where

$$P[w] = P(t) \equiv \rho\int_{-l_2}^{l_1} wdx + m_1w_1 + m_2w_2$$
$$= \rho\int_{-l_2}^{l_1} wdx + m_1\left[w + a_1\frac{\partial w}{\partial x}\right]_{x=l_1} + m_2\left[w = a_2\frac{\partial w}{\partial x}\right]_{x=-l_2} \tag{7}$$

Equations (5) and (6) mathematically locate the origin of the moving coordinate system; however, a criterion establishing its angular position corresponding to a given configuration of the system has not been set. The expression for the angular momentum $\mathbf{h}$ of the system will be examined since this expression indicates the choice of a convenient criterion. $\mathbf{h}$ is in the $\mathbf{k}$ direction and within linear terms its magnitude is

$$h = J\dot{\theta} + dQ(t)/dt \tag{8}$$

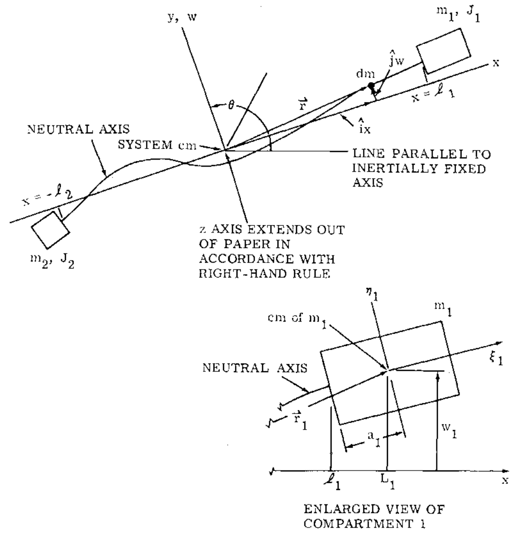

Fig. 1 System diagram.

In the preceding, J is the mass moment of inertia of the system about the z axis when the deformation $w = 0$; i.e.,

$$J \equiv \int_{UV} (x^2 + y^2)dm = \frac{1}{3}\rho(l_1^3 + l_2^3) + \Gamma(l_1 + l_2) + m_1L_1^2 + m_2L_2^2 + J_{\zeta_1} + J_{\zeta_2} \tag{9}$$

where UV indicates that the integration is performed over the undeformed vehicle, and $Q(t)$ is defined as follows:

$$Q[w] = Q[t] \equiv \rho\int_{-l_2}^{l_1} wxdx + \Gamma w\Big|_{-l_2}^{l_1} + m_1L_1w(l_1,t) - m_2L_2w(-l_2,t) + (m_1a_1L_1 + J_{\zeta_1})\frac{\partial w(l_1,t)}{\partial x} + (m_2a_2L_2 + J_{\zeta_2})\frac{\partial w(-l_2,t)}{\partial x} \tag{10}$$

Now, a condition is sought to locate the angular position of the x,y axes with respect to the system. From Fig. 1 this condition will also determine the value of $\theta(t)$. As indicated in Ref. 5, any orientation of the axes is permissible provided they stay close to the system so that linearization is possible. For the case of a free system there is no external torque about the c.m.; therefore h is constant. By imposing the condition

$$Q[w] = 0 \tag{11}$$

on the deflection curve, $\dot{\theta}$ is constant in accordance with Eq. (8), and, accordingly, a simplification is obtained for the free-problem formulation. Equation (11) prescribes the angular position of the axes with respect to the deflection curve such that an optimum fit to the curve is achieved in accordance with a certain criterion.[7] It is seen from Eq. (8) that for the forced as well as the free case, the axes, as oriented by $Q = 0$, rotate with the system at its average angular velocity, $\dot{\theta} = h/J$. The conditions $P[w] = 0$ [which, along with Eq. (5), served to locate the origin] and $Q[w] = 0$ locate the axes in the neighborhood of the deflection curve and will be referred to as the coordinate conditions. The use of this type of coordinate system has been discussed by Milne.[8] In fact, for linearized systems these coordinates are a special case of the mean axes of Tisserand.[9] It is noted that for the free nonrotating system where $\dot{\theta} = 0$, Eq. (8) shows $Q[w] = 0$ is the usual condition imposed on a semidefinite system to insure the vanishing of the angular momen-

‡ For additional details of the analysis (such as intermediate manipulations, comparisons of limiting cases with results of other investigators, etc.) see Ref. 7.

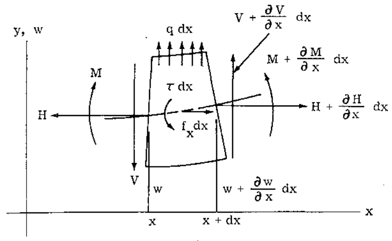

Fig. 2 Free-body diagram of beam element.

tum; also Eq. (6), $P[w] = 0$, turns out to be the usual condition imposed to insure vanishing of the linear momentum.

The equation of motion of the beam is formulated in the classical way; i.e., the three planar equations of motion for the free-body differential beam element shown in Fig. 2 are combined. In this process only linear terms are retained. The influence of rotatory inertia is included; however transverse shear is neglected. The resulting differential equation (DE) is

$$EI\frac{\partial^4 w}{\partial x^4} - \frac{\partial}{\partial x}\left\{\left[B_{1x} - [m_1 + \rho(l_1 - x)]a_{cx} + \int_x^{l_1} f_x dx + \dot{\theta}^2[m_1L_1 + \frac{1}{2}\rho(l_1^2 - x^2)]\right]\frac{\partial w}{\partial x}\right\} = q - \rho\left(a_{cy} + \frac{\partial^2 w}{\partial t^2} - \dot{\theta}^2 w + \ddot{\theta}x\right) - \frac{\partial \tau}{\partial x} + \Gamma\frac{\partial^4 w}{\partial t^2 \partial x^2} \quad (12)$$

Figure 3 shows the free-body diagrams of both tip masses. In order to obtain the boundary conditions (BCs), the equations of motion for the tip masses are combined, and the results are

BCs:

$$EI\frac{\partial^2 w}{\partial x^2} + a_1\left[B_{1x} - m_1(a_{cx} - l_1\dot{\theta}^2)\right]\frac{\partial w}{\partial x} - m_1a_1\dot{\theta}^2 w = -m_1a_1\frac{\partial^2 w}{\partial t^2} - (m_1a_1^2 + J_{\zeta_1})\frac{\partial^3 w}{\partial t^2\partial x} - (m_1a_1L_1 + J_{\zeta_1})\ddot{\theta} + T_1 + a_1(B_{1y} - m_1a_{cy}) \quad \text{at } x = l_1 \quad (13)$$

$$EI\frac{\partial^3 w}{\partial x^3} - [B_{1x} - m_1(a_{cx} - l_1\dot{\theta}^2)]\frac{\partial w}{\partial x} + m_1\dot{\theta}^2 w = m_1\frac{\partial^2 w}{\partial t^2} + (m_1a_1 + \Gamma)\frac{\partial^3 w}{\partial t^2\partial x} + (m_1L_1 + \Gamma)\ddot{\theta} + m_1a_{cy} - \tau - B_{1y} \quad \text{at } x = l_1 \quad (14)$$

$$EI\frac{\partial^2 w}{\partial x^2} + a_2[B_{2x} - m_2(a_{cx} + l_2\dot{\theta}^2)]\frac{\partial w}{\partial x} - m_2a_2\dot{\theta}^2 w = -m_2a_2\frac{\partial^2 w}{\partial t^2} + (m_2a_2^2 + J_{\zeta_2})\frac{\partial^3 w}{\partial t^2\partial x} + (m_2a_2L_2 + J_{\zeta_2})\ddot{\theta} - T_2 + a_2(B_{2y} - m_2a_{cy}) \quad \text{at } x = -l_2 \quad (15)$$

$$EI\frac{\partial^3 w}{\partial x^3} + [B_{2x} - m_2(a_{cx} + l_2\dot{\theta}^2)]\frac{\partial w}{\partial x} - m_2\dot{\theta}^2 w = -m_2\frac{\partial^2 w}{\partial t^2} + (m_2a_2 + \Gamma)\frac{\partial^3 w}{\partial t^2\partial x} + (m_2L_2 + \Gamma)\ddot{\theta} - m_2a_{cy} - \tau + B_{2y} \quad \text{at } x = -l_2 \quad (16)$$

In order to solve the formulated problem, expressions are required for the inertial components of the system c.m., $a_{cx}(t)$ and $a_{cy}(t)$, and the angular velocity of the axes, $\dot{\theta}(t)$. Since the origin of the axes coincides with the system c.m., these terms describe the motion of the axes. $\dot{\theta}$ is not arbitrary since the axes are required to satisfy the coordinate condition $Q[w] = 0$.

The additional equations used to obtain these terms are the over-all equations of motion for the system. For the x and y directions, the equations are

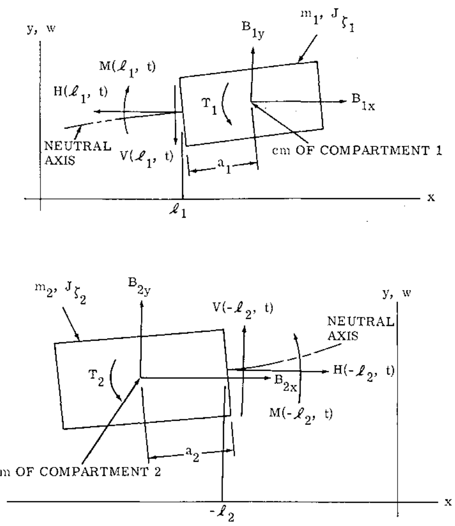

Fig. 3 Free-body diagrams of the end compartments.

$$\int_{-l_2}^{l_1} f_x dx + B_{1x} + B_{2x} = M_s a_{cx} \quad (17)$$

$$\int_{-l_2}^{l_1} q dx + B_{1y} + B_{2y} = M_s a_{cy} \quad (18)$$

where

$$M_s \equiv \rho(l_1 + l_2) + m_1 + m_2 \quad (19)$$

To obtain the torque equation, it is first noted that the angular momentum about the system c.m., Eq. (8), with the coordinate condition $Q(t) = 0$ becomes

$$h = J\dot{\theta} \quad (20)$$

Since J is the constant moment of inertia of the undeformed system, the net torque about the vehicle c.m. is

$$T_c = J\ddot{\theta} \quad (21)$$

Equations (20) and (21) are the usual relations for rigid bodies. As a consequence of the coordinate condition used above, these equations are also applicable for the elastic system under consideration. Expanding T_c, Eq. (21) becomes

$$\int_{-l_2}^{l_1} (qx - f_x w + \tau)dx - B_{1x}w_1 - B_{2x}w_2 + B_{1y}L_1 - B_{2y}L_2 + T_1 + T_2 = J\ddot{\theta} \quad (22)$$

If the forces are prespecified functions of t, a_{cx} and a_{cy} could be obtained from Eqs. (17) and (18) and the results could be used in the elastic Equations, (12–16); however, unless the horizontal externally applied forces, B_{1x} and B_{2x}, and the force density, f_x, vanished, the equation for $\ddot{\theta}$, (22), is coupled with the elastic equations. Similar coupling occurs for a_{cx} and a_{cy} as well as for $\ddot{\theta}$ when the forcing terms are functions of the deflection, w.

Equation (21) also shows that the angular velocity of the axes used here may only be considered constant when the resultant torque about the system c.m. vanishes identically.

Some Fundamental Properties of the Free System

For the case of the free system the acceleration of the c.m. vanishes, and, in accordance with the developments of the previous section, the angular velocity of the axes, $\omega \equiv \dot{\theta}$, is constant. Setting the forcing terms in Eqs. (12–16) to zero yields the following formulation:

DE:

$$EI\,\frac{\partial^4 w}{\partial x^4} - \omega^2\,\frac{\partial}{\partial x}\left\{\left[m_1L_1 + \frac{1}{2}\,\rho(l_1^2 - x^2)\right]\frac{\partial w}{\partial x}\right\} - \omega^2\rho w = -\rho\,\frac{\partial^2 w}{\partial t^2} + \Gamma\,\frac{\partial^4 w}{\partial x^2\partial t^2} \qquad (23)$$

BCs:

$$EI\,\frac{\partial^2 w}{\partial x^2} + m_1a_1\omega^2\left(l_1\frac{\partial w}{\partial x} - w\right) = -m_1a_1\,\frac{\partial^2 w}{\partial t^2} - (J_{\zeta_2} + m_1a_1^2)\,\frac{\partial^3 w}{\partial t^2\partial x} \quad \text{at } x = l_1 \qquad (24)$$

$$EI\,\frac{\partial^3 w}{\partial x^3} - m_1\omega^2\left(l_1\frac{\partial w}{\partial x} - w\right) = m_1\,\frac{\partial^2 w}{\partial t^2} + (\Gamma + m_1a_1)\,\frac{\partial^3 w}{\partial t^2\partial x} \quad \text{at } x = l_1 \qquad (25)$$

$$EI\,\frac{\partial^2 w}{\partial x^2} - m_2a_2\omega^2\left(l_2\frac{\partial w}{\partial x} + w\right) = -m_2a_2\,\frac{\partial^2 w}{\partial t^2} + (J_{\zeta_2} + m_2a_2^2)\,\frac{\partial^3 w}{\partial x\partial t^2} \quad \text{at } x = -l_2 \qquad (26)$$

$$EI\,\frac{\partial^3 w}{\partial x^3} - m_2\omega^2\left(l_2\frac{\partial w}{\partial x} + w\right) = -m_2\,\frac{\partial^2 w}{\partial t^2} + (\Gamma + m_2a_2)\,\frac{\partial^3 w}{\partial t^2\partial x} \quad \text{at } x = -l_2 \qquad (27)$$

The variables are separated by assuming a harmonic solution;

$$w(x,t) = v(x)e^{ipt} \qquad (28)$$

By substitution of Eq. (28) into Eqs. (23–27), the following differential equation and boundary conditions are obtained:
DE:

$$G(v) = p^2N(v) \qquad (29)$$

where G and N are the differential operators,

$$G(v) \equiv EIv^{iv} - \omega^2\{[m_1L_1 + \tfrac{1}{2}\rho(l_1^2 - x^2)]v'\}' - \omega^2\rho v \qquad (30)$$

$$N(v) \equiv -\Gamma v'' + \rho v \qquad (31)$$

BCs:

$$EIv'' + m_1a_1\omega^2(l_1v' - v) = p^2[(J_{\zeta_1} + m_1a_1^2)v' + m_1a_1v] \text{ at } x = l_1 \qquad (32)$$

$$EIv''' - m_1\omega^2(l_1v' - v) = -p^2[(\Gamma + m_1a_1)v' + m_1v] \text{ at } x = l_1 \qquad (33)$$

$$EIv'' - m_2a_2\omega^2(l_2v' + v) = p^2[-(J_{\zeta_2} + m_2a_2^2)v' + m_2a_2v] \text{ at } x = -l_2 \qquad (34)$$

$$EIv''' - m_2\omega^2(l_2v' + v) = p^2[-(\Gamma + m_2a_2)v' + m_2v] \text{ at } x = -l_2 \qquad (35)$$

Orthogonality Conditions and Reality of Eigenvalues

Each eigenfunction, v_n, and its corresponding eigenvalue, p_n^2, satisfies the formulation of Eqs. (29–35). Integration by parts yields

$$\int_{-l_2}^{l_1} v_jN(v_n)dx = \int_{-l_2}^{l_1} (\Gamma v_j'v_n' + \rho v_jv_n)dx - \Gamma v_jv_n'\Big|_{-l_2}^{l_1} \qquad (36)$$

thus

$$\int_{-l_2}^{l_1} [v_jN(v_n) - v_nN(v_j)]dx = \Gamma(v_nv_j' - v_jv_n')\Big|_{-l_2}^{l_1}$$

and this quantity does not vanish. From this aspect alone, the problem is not self-adjoint in the ordinary sense.[10] Orthogonality conditions can, however, be established in the following way. From Eq. (29),

$$\int_{-l}^{l_1} [v_jG(v_n) - v_nG(v_j)]dx = \int_{-l}^{l} [p_n^2v_jN(v_n) - p_j^2v_nN(v_j)]dx \qquad (37)$$

Integration by parts, using Eq. (5) and Eqs. (32–36), yields the following orthogonality condition:

$$\langle v_j,v_n\rangle = 0;\ \ j \neq n, \text{ and } \langle v_n,v_n\rangle = N_n \qquad (38\text{a})\S$$

where

$$\langle v_j,v_n\rangle \equiv \int_{-l_2}^{l_1} (\Gamma v_j'v_n' + \rho v_jv_n)dx + [J_{\zeta_1}v_j'v_n' + m_1(a_1v_j' + v_j)(a_1v_n' + v_n)]|_{x=l_1} + [J_{\zeta_2}v_j'v_n' + m_2(a_2v_j' - v_j)(a_2v_n' - v_n)]|_{x=-l_2} \qquad (39)$$

The following alternative relation to Eq. (38a) is obtained by substituting Eq. (36):

$$\langle v_j,v_n\rangle = \int_{-l_2}^{l_1} v_jN(v_n)dx + [J_{\zeta_1}v_j'v_n' + m_1(a_1v_j' + v_j)(a_1v_n' + v_n)]|_{x=l_1} + [J_{\zeta_2}v_j'v_n' + m_2(a_2v_j' - v_j)(a_2v_n' - v_n)]|_{x=-l_2} + \Gamma v_jv_n'\Big|_{-l_2}^{l_1} = \begin{cases} 0; & j \neq n \\ N_n; & j = n \end{cases} \qquad (38\text{b})$$

The second orthogonality condition is obtained by multiplying the preceding equation throughout by p_n^2 and substituting Eqs. (29 and 32–35). This yields

$$\int_{-l_2}^{l_1} v_jG(v_n)dx + EI(-v_jv_n''' + v_j'v_n'')\Big|_{-l_2}^{l_1} + m_1\omega^2(a_1v_j' + v_j)(l_1v_n' - v_n)|_{x=l_1} + m_2\omega^2(a_2v_j' - v_j)(l_2v_n' + v_n)|_{x=-l_2} = \begin{cases} 0; & j \neq n \\ p_n^2N_n; & j = n \end{cases} \qquad (40\text{a})$$

or, alternatively,

$$\int_{-l_2}^{l_1} \{EIv_j''v_n'' + \omega^2[m_1L_1 + \frac{1}{2}\,\rho(l_1^2 - x^2)]v_j'v_n' - \rho\omega^2v_jv_n\}dx + m_1\omega^2[a_1l_1v_j'v_n' - a_1(v_jv_n' + v_nv_j') - v_jv_n]|_{x=l_1} + m_2\omega^2[a_2l_2v_j'v_n' + a_2(v_jv_n' + v_nv_j') - v_jv_n]|_{x=-l_2} = \begin{cases} 0; & j \neq n \\ p_n^2N_n; & j = n \end{cases} \qquad (40\text{b})$$

The fact that all of the eigenvalues, the p_n^2's, are real will now be established. Since the coefficients in the formulation, Eqs. (29–35), are real, if v_n and p_n^2 are the corresponding eigenfunction and eigenvalue, $\bar{v}_n$ is also an eigenfunction with $\bar{p}_n^2$ as its eigenvalue.¶ If p_n^2 were complex, $p_n^2 \neq \bar{p}_n^2$, and Eq. (38a) would imply $\langle \bar{v}_n,v_n\rangle = 0$. However, v_n is an eigenfunction; consequently $\bar{v}_n \not\equiv 0$ for $-l_2 \leq x \leq l_1$, and Eq. (39) gives rise to the contradiction $\langle \bar{v}_n,v_n\rangle > 0$. Thus, $p_n^2 = \bar{p}_n^2$; i.e., p_n^2 is real.

Since all of the p_n^2 are real, it also follows that v_n and $\bar{v}_n$ have the same eigenvalue. In consequence, any linear combination of v_n and $\bar{v}_n$ is an eigenfunction with the eigenvalue p_n^2; thus the real and imaginary parts of v_n can be used as eigenfunctions instead of v_n and $\bar{v}_n$, and these quantities

§ In the event that two or more linearly independent eigenfunctions possess the same eigenvalue, they can be orthogonalized by the Schmidt process[11]; thus all of the linearly independent eigenfunctions have been taken as orthogonal.

¶ The bar indicates the complex conjugate.

will be linearly independent if v_n and $\bar{v}_n$ are linearly independent. Also, by suitably normalizing the eigenfunctions, the constants of proportionality will be real; therefore, a set of linearly independent eigenfunctions, in which each eigenfunction is a real quantity, can be considered.

Finally, setting $j = n$ in Eqs. (38a) and (40b), eliminating N_n, and rearranging yields

$$p_n^2 = \frac{\int_{-l_2}^{l_1} EI(v_n'')^2 dx + \omega^2 S[v_n]}{\langle v_n, v_n \rangle} - \omega^2 \qquad (41)$$

where

$$S[v_n] = \int_{-l_2}^{l_1} [m_1 L_1 + \Gamma + \frac{1}{2}\rho(l_1^2 - x^2)](v_n')^2 dx +$$
$$(J_{\zeta_1} + m_1 a_1 L_1)(v_n')^2|_{x=l_1} + (J_{\zeta_2} + m_2 a_2 L_2)(v_n')^2|_{x=-l_2} \qquad (42)$$

It is seen that $p_n^2 \geq -\omega^2$; thus all eigenvalues are bounded from below. The problem (29–35) possesses the following simple solutions:

$$v = 1; \quad p^2 = -\omega^2 \qquad (43)$$

$$v = x; \quad p^2 = 0 \qquad (44)$$

Thus, one of the eigenvalues is actually equal to the lower bound, $-\omega^2$. The solutions given by Eqs. (43) and (44) are attributable to rigid-body displacements of the system and are, therefore, called the rigid-body modes as opposed to the elastic or bending modes. Since the system is conservative, the remaining eigenvalues, which are the squares of the frequencies for the bending modes, must all be positive.

Relations Among the Eigenfunctions and the Coordinate Conditions

In view of Eqs. (43) and (44), w has the form

$$w(x,t) = C_a e^{\omega t} + C_{-a} e^{-\omega t} + x(C_b + C_b' t) + \sum_{j=1}^{\infty} v_j(x)[C_j e^{ip_j t} + C_{-j} e^{-ip_j t}] \qquad (45)$$

where the C's are constants and the summation is taken over all eigenfunctions other than those of Eqs. (43) and (44). It is seen that the rigid-body modes are associated with time-increasing terms in Eq. (45); however the appearance of increasing rigid-body deflections is inconsistent with the geometrical interpretation of the coordinate conditions. This situation is investigated as follows. By allowing each of the rigid-body modes, $v = 1$ and $v = x$, in turn, to take on the value v_j in definition Eq. (39) and comparing the resulting expressions with Eqs. (7) and (10), it is determined that

$$\langle 1, v_n \rangle = P[v_n] \qquad (46)$$

and

$$\langle x, v_n \rangle = Q[v_n] \qquad (47)$$

Using the orthogonality condition Eq. (38a) along with the preceding relations,

$$P[v_n] = 0, \ v_n \neq K_1 \qquad (48$$

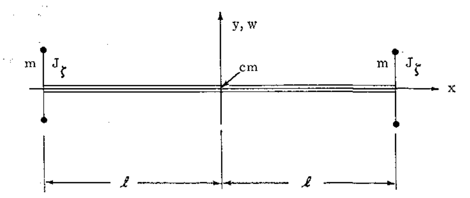

Fig. 4 The H-type space station.

and

$$Q[v_n] = 0, \ v_n \neq K_2 x \qquad (49)$$

where K_1 and K_2 are arbitrary constants. Thus, it is seen that any eigenfunction other than a constant automatically satisfies the first coordinate condition, $P[v_n] = 0$, and any eigenfunction which is not proportional to the rigid-body mode x automatically satisfies the second coordinate condition, $Q[v_n] = 0$. Noting that P and Q are linear functionals and using Eq. (45) with Eqs. (48) and (49), the coordinate conditions yield

$$P[w(x,t)] = (C_a e^{\omega t} + C_{-a} e^{-\omega t})P[1] = 0$$

and

$$Q[w(x,t)] = (C_b + C_b' t)Q[x] = 0$$

Therefore, $C_a P[1] = C_{-a} P[1] = C_b Q[x] = C_b' Q[x] = 0$. Applying the definitions of Eqs. (7) and (10), $P[1] \neq 0$ and $Q[x] \neq 0$; thus, $C_a = C_{-a} = C_b = C_{-b} = 0$. In consequence, if w satisfies the coordinate conditions, $P[w] = Q[w] = 0$, the eigenfunction expansion for w, Eq. (45), cannot contain the rigid-body modes, $v = 1$ and $v = x$. Conversely, if all eigen functions in the expansion for w obey the equations $P[v_n] = Q[v_n] = 0$, the coordinate conditions, $P[w] = Q[w] = 0$, are satisfied. From these facts it is easily established that all of the items on the following list are equivalent coordinate conditions. The coordinates are selected such that:

1) $P[w] = Q[w] = 0$.

2) $P[v_n] = Q[v_n] = 0$ for all modes v_n which appear in the eigenfunction expansion for w.

3) All modes which appear in the eigenfunction expansion for w are orthogonal to the rigid-body modes, $v = 1$ and $v = x$.

4) No terms corresponding to the rigid-body modes, $v = 1$ and $v = x$, appear in the eigenfunction expansion for w.

In the same way it can be shown that the above conditions are equivalent for the case of forced response where an eigenfunction expansion is assumed with general time-varying coefficients.

Development of Normal Modes and Natural Frequencies for an H-Type Space Station

Aside from the simple modes which are expressed by Eqs. (43) and (44), the problem (29–35) apparently has no known exact solution. In this paper numerical values for the normal modes and natural frequencies will be developed for one configuration, called the H-type space station, by employing the Rayleigh-Ritz method.

In the H-type space station most of the equipment is assumed to be located in the end compartments; thus the beam is primarily a structural member, and its rotatory inertia coefficient, Γ, is neglected. A practical example of an early space station of this sort would be the final stages of two rockets which are connected by a light beam. In order to simplify the analysis and still retain the ability to ascertain the influence of the rotatory inertia of the compartments, the system is assumed symmetrical, and the dimension a_1, the distance between the compartment c.m. and beam-compartment connection point, is assumed to be zero. The idealized configuration resembles an elongated H and is indicated in Figure 4. For this case,

$$\left.\begin{aligned} \Gamma &= 0, \ m \equiv m_1 = m_2, \ J_\zeta \equiv J_{\zeta_1} = J_{\zeta_2} \\ a_1 &= a_2 = 0, \ l \equiv L_1 = L_2 = l_1 = l_2 \end{aligned}\right\} \qquad (50)$$

The problem is nondimensionalized by use of the following definitions:

$$\left.\begin{aligned} & u_n \equiv v_n/l, \ s \equiv x/l, \ \epsilon \equiv (\rho l^4/EI)\omega^2, \ r \equiv m/\rho l \\ & \mu \equiv J_\zeta/ml^2, \ \lambda_n \equiv (\rho l^4/EI)p_n^2, \ \tilde{N}_n \equiv N_n/\rho l^3 \\ & \tilde{G}(u_n) \equiv (l^3/EI)G(v_n) \end{aligned}\right\} \qquad (51)$$

The formulation becomes (with a prime now indicating differentiation with respect to s),

DE:

$$\tilde{G}(u_n) = u_n^{iv} - \epsilon\{[r + \tfrac{1}{2}(1 - s^2)]u_n'\}' - \epsilon u_n = \lambda_n u_n \quad (52)$$

BCs:

$$u_n'' = r\mu\lambda_n u_n' \quad \text{at } s = 1 \quad (53)$$

$$u_n''' - \epsilon r u_n' + \epsilon r u_n = -r\lambda_n u_n \quad \text{at } s = 1 \quad (54)$$

$$u_n'' = -r\mu\lambda_n u_n' \quad \text{at } s = -1 \quad (55)$$

$$u_n''' - \epsilon r u_n' - \epsilon r u_n = r\lambda_n u_n \quad \text{at } s = -1 \quad (56)$$

In consequence of the symmetry of the problem, the normal modes can be expressed as either even or odd functions of x.

In the following the normal modes and natural frequencies will be obtained by the Rayleigh-Ritz method. For use in this work a function will be called admissible if it has continuous derivatives up to and including the fourth in the closed interval $-1 \leq s \leq 1$. Only real functions are considered, and the class of all real admissible functions is called $C^{(4)}$. Let

$$X[y,z] \equiv \int_{-1}^{1} yzds + r(\mu y'z' + yz)|_{s=1} + r(\mu y'z' + yz)|_{s=-1} \quad (57)$$

and

$$Z[y,z] \equiv \int_{-1}^{1} y\tilde{G}(z)ds - y(z''' - \epsilon r z' + \epsilon r z)|_{s=1} + y(z''' - \epsilon r z' - \epsilon r z)|_{s=-1} + y'z''\Big|_{-1}^{1} \quad (58)$$

where $y(s)$ and $z(s)$ are any admissible functions. Also, the notation, $X[y] \equiv X[y,y]$ and $Z[y] \equiv Z[y,y]$, will be used. By integration by parts,

$$Z[y,z] = Z[z,y] = \int_{-1}^{1} \{y''z'' + \epsilon[r + \frac{1}{2}(1 - s^2)]y'z' - \epsilon yz\}ds - \epsilon r[y(1)z(1) + y(-1)z(-1)] \quad (59)$$

It is seen that X and Z are bilinear functionals. X is positive definite, and it possesses the properties of a scalar product for the elements of $C^{(4)}$.

The orthogonality conditions, Eqs. (38a) and (40b), are written in nondimensional form and may be stated as follows:

$$X[u_j,u_n] = 0; \quad j \neq n, \text{ and } X[u_n,u_n] = \tilde{N}_n \quad (60)$$

$$Z[u_j,u_n] = 0; \quad j \neq n, \text{ and } Z[u_n,u_n] = \lambda_n\tilde{N}_n \quad (61)$$

Also, the coordinate conditions which express the orthogonality of all eigenfunctions which are permitted to appear in the eigenfunction expansion (i.e., all of the bending modes) to the rigid-body modes, $u = 1$ with $\lambda = -\epsilon$ and $u = s$ with $\lambda = 0$, become

$$X[1,u_n] = X[s,u_n] = 0. \quad (62)$$

By using standard techniques of the calculus of variations, it can be shown that the quantity,

$$D[y] \equiv Z[y]/X[y] \quad (63)$$

has the properties of Rayleigh's quotient; that is, for all admissible functions $y(s)$, the Rayleigh quotient, $D[y]$, is stationary if and only if y is an eigenfunction. The stationary value, $D[u_n]$, is the corresponding eigenvalue, λ_n.

The usual proof of the Rayleigh-Ritz method[12] is limited to self-adjoint systems of the ordinary type. A similar proof is developed for the present formulation in Ref. 7, and the results are as follows: A set of N linearly independent functions $h_i(s)$ $(i = 1,2, \ldots, N)$ are selected which are elements of $C^{(4)}$ and are orthogonal to the rigid-body modes $u = 1$ and $u = s$ (i.e., $X[h_i,1] = X[h_i,s] = 0$). Then, the first N elastic modes u_i $(i = 1, \ldots, N)$ are approximated by functions of the form

$$y_n = \sum_{j=1}^{N} E_j^{(n)}h_j(s) \quad (64)$$

where $E_j^{(n)}(j = 1, \ldots, N)$ is given by the nth eigenvector of the problem

$$\sum_{j=1}^{N} (Z[h_i,h_j] - \Lambda X[h_i,h_j])E_j = 0; \quad j = 1,2, \ldots, N \quad (65)$$

and the nth eigenvalue λ_n is approximated from above by the corresponding nth eigenvalue Λ_n. Furthermore, by increasing the number of coordinate functions, N, the approximations to the initial N eigenvalues λ_n $(n = 1,2, \ldots, N)$ are improved, or at least not worsened.

The normal bending modes of the nonrotating system u_{jo} $(j = 1,2, \ldots, N)$ are now selected as the coordinate functions; i.e., $h_j = u_{jo}$ $(j = 1,2, \ldots, N)$. It will be seen that these coordinate functions have the advantage of being orthogonal with respect to the inner product X thereby effecting a considerable simplification in the computation procedure, and they obey the coordinate conditions; i.e., the u_{jo}'s are exactly orthogonal to the rigid-body modes $u = 1$ and $u = s$.

The formulation, orthogonality, and coordinate conditions for the nonrotating problem are easily obtained by setting $\epsilon = 0$ in the previous results. Since ϵ does not appear in the definition of X, Eq. (57), it is seen that the eigenfunctions for the nonrotating problem obey the first orthogonality condition for the rotating problem, Eq. (60), exactly; i.e.,

$$X[u_{jo},u_{no}] = 0; \quad \lambda_{jo} \neq \lambda_{no}, \text{ and } X[u_{jo},u_{no}] = 1 \quad (66)$$

where the eigenfunctions have been normalized such that $X[u_{no}] = 1$. The second orthogonality condition is, from Eqs. (58), (59), and (61),

$$\int_{-1}^{1} u_{jo}''u_{no}''ds = \int_{-1}^{1} u_{jo}\tilde{G}_o(u_{no})ds - (u_{jo}u_{no}''' - u_{jo}'u_{no}'')\Big|_{-1}^{1} = \begin{cases} 0; & \lambda_{jo} \neq \lambda_{no}, \\ \lambda_{no}; & \lambda_{jo} = \lambda_{no}. \end{cases} \quad (67)$$

The coordinate conditions, Eqs. (62), are obeyed exactly by the bending modes of the nonrotating problem; i.e., $X[(1),u_{no}] = X[s,u_{no}] = 0$. From Eqs. (59) and (61)

$$\lambda_{no} = \int_{-1}^{1} u_{no}''^2 ds$$

which is greater than zero except for the zero-frequency rigid-body modes $u = 1$ and $u = s$.

The solutions for the bending modes are obtained in the usual way. They are expressed in terms of

$$\beta_n \equiv \lambda_{no}^{1/4} > 0 \quad (68)$$

The solutions follow.

Even modes ($n = 1,3, \ldots$): When β_n satisfies the characteristic equation,

$$(1 - \mu r^2\beta_n^4)(\sin\beta_n + \tanh\beta_n \cos\beta_n) + 2r\beta_n(\cos\beta_n - \mu\beta_n^2 \tanh\beta_n \sin\beta_n) = 0 \quad (69)$$

the mode shape is the even function,

$$u_{no}(s) = C_n(\cos\beta_n s - Y_n \cosh\beta_n s) \quad (70)$$

where

$$Y_n \equiv \frac{\sin\beta_n + r\beta_n \cos\beta_n}{\sinh\beta_n + r\beta_n \cosh\beta_n} \quad (71)$$

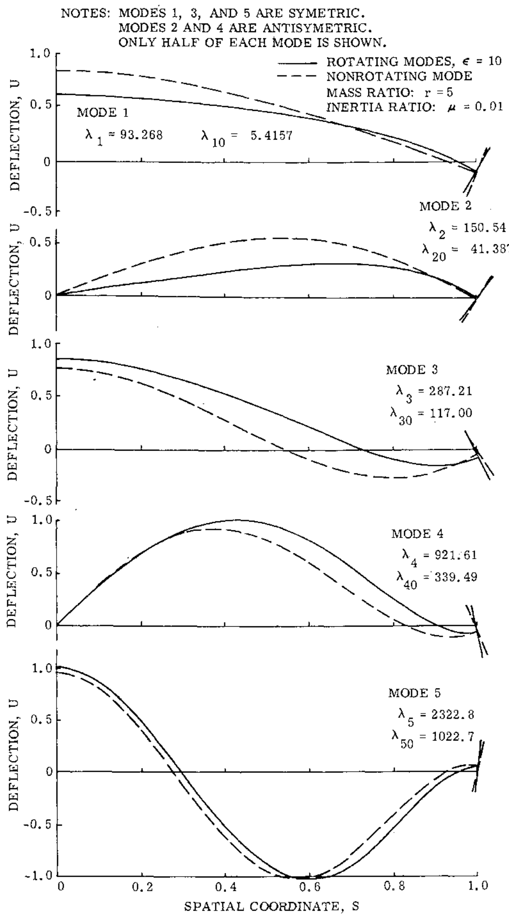

Fig. 5 First five rotating and nonrotating modes.

and

$$C_n^{-2} = W_E(\beta_n,\beta_n, -Y_n, -Y_n) + 2r[\mu\beta_n^2(\sin\beta_n + Y_n \sinh\beta_n)^2 + (\cos\beta_n - Y_n \cosh\beta_n)^2] \quad (72)^{**}$$

Odd modes ($n = 2,4, \ldots$): When β_n satisfies the characteristic equation,

$$(1 - \mu r^2\beta_n^4)(\sin\beta_n - \cos\beta_n \tanh\beta_n) + 2r\beta_n(\mu\beta_n^2 \cos\beta_n + \sin\beta_n \tanh\beta_n) = 0 \quad (73)$$

the mode shape is the odd function,

$$u_{no}(s) = K_n(\sin\beta_n s + Z_n \sinh\beta_n s) \quad (74)$$

where

$$Z_n \equiv \frac{\cos\beta_n - r\beta_n \sin\beta_n}{\cosh\beta_n + r\beta_n \sinh\beta_n} \quad (75)$$

and

$$K_n^{-2} = W_0(\beta_n,\beta_n,Z_n,Z_n) + 2r[\mu\beta_n^2(\cos\beta_n + Z_n \cosh\beta_n)^2 + (\sin\beta_n + Z_n \sinh\beta_n)^2] \quad (76)$$

These results will now be used in the Rayleigh-Ritz procedure to obtain the normal modes and natural frequencies for the rotating system. In the following all results are the Rayleigh-Ritz approximations; therefore the use of the symbols $u_j(s)$ and λ_j, previously reserved for the exact solutions, will not cause confusion. From Eq. (64) the nth bending mode is

$$u_n(s) = \sum_{j=1}^{N} E_j^{(n)}u_{jo}(s) \quad (77)$$

With the assistance of Eq. (66), Eq. (65) for the eigenvalue λ_n and the components $E_j^{(n)}$ of the eigenvector u_n ($j = 1,2\ldots, N$) is written in matrix form as follows:

$$([Z] - \lambda_n[I])\{E^{(n)}\} = 0 \quad (78)$$

where $\{E^{(n)}\} = [E_1^{(n)}E_2^{(n)} \ldots E_N^{(n)}]^T$, and $[Z]$ has the components,

$$z_{ij} \equiv Z[u_{io},u_{jo}] \quad (79)$$

During the proof of the Rayleigh-Ritz procedure,[7] it was determined that the approximate eigenfunctions obey the same orthogonality conditions Eqs. (60) and (61), as the exact eigenfunctions; i.e.,

$$X[u_i,u_j] = \begin{cases} 1; & i = j \\ 0; & i \pm j \end{cases} \text{ and } Z[u_i,u_j] = \begin{cases} 0\ ; & i \neq j \\ \lambda_j; & i = j \end{cases} \quad (80)$$

where the eigenfunctions are normalized such that $X[u_n] = 1$. By substituting Eq. (77) and using Eq. (66), this condition yields

$$\sum_{j=1}^{N} (E_j^{(n)})^2 = 1 \quad (81)$$

Table 1 Variation of frequency with angular velocity for system with larger tip mass (r = 20, μ = 0.1)

Angular velocity parameter	Frequency parameters				
ϵ	λ_1	λ_2	λ_3	λ_4	λ_5
0	5.416	41.39	117.0	339.5	1023
5	53.24	107.8	208.6	628.2	1672
10	93.27	150.5	287.2	921.6	2323
15	127.5	183.8	362.0	1214	2971
20	157.4	211.7	435.7	1504	3616
25	184.0	236.4	508.9	1793	4258
30	208.0	258.6	582.0	2081	4898
35	230.0	279.0	655.0	2367	5535
40	250.4	298.1	728.1	2653	6171
45	269.5	316.0	801.1	2938	6805
50	287.4	332.9	874.2	3222	7438

Table 2 Variation of frequency with angular velocity (r = 20, μ = 0.1)

Angular velocity parameter	Frequency parameters				
ϵ	λ_1	λ_2	λ_3	λ_4	λ_5
0	0.4695	1.620	34.35	242.8	921.6
5	4.968	6.087	324.0	1,367	3,399
10	7.061	8.362	599.3	2,452	5,820
15	8.688	10.14	870.4	3,521	8,216
20	10.09	11.66	1139	4,583	10,600
25	11.35	13.03	1407	5,640	12,970
30	12.51	14.30	1674	6,693	15,330
35	13.61	15.49	1940	7,744	17,690
40	14.66	16.63	2205	8,793	20,050
45	15.67	17.73	2470	9,840	22,400
50	16.66	18.79	2735	10,890	24,750

** The function W_E, as well as the functions W_0, W_{E2}, and W_{02} which will appear shortly, are integrals involving transcendental functions which appear often in the numerical work. These integrals are defined in the Appendix.

For the purposes of computation the elements z_{ij} of $[Z]$ are expanded in accordance with Eq. (59). Equations (67) and (70) or (74) are substituted into the resulting expression. $z_{ij} = 0$ when one mode is even and the other is odd, and, because of this, $E_k^{(n)} = 0$ when k designates an even mode and n an odd mode, or vice versa. Otherwise,

$$z_{ij} = \delta_{ij}\lambda_{jo} + \epsilon[(r + \tfrac{1}{2})\beta_i\beta_j C_i C_j W_0(\beta_i,\beta_j,Y_i,Y_j) - \tfrac{1}{2}\beta_i\beta_j C_i C_j W_{02}(\beta_i,\beta_j,Y_i,Y_j) - C_i C_j W_E(\beta_i,\beta_j,-Y_i,-Y_j) - 2ru_{io}(1)u_{jo}(1)] \quad (82)$$

when u_{io} and u_{jo} are even where $u_{i0}(1)$ and $u_{jo}(1)$ are given by Eq. (70), W_0, W_{02}, and W_E are integrals which are defined in the Appendix, and

$$z_{ij} = \delta_{ij}\lambda_{j0} + \epsilon[(r + \tfrac{1}{2})\beta_i\beta_j K_i K_j W_E(\beta_i,\beta_j,Z_i,Z_j) - \tfrac{1}{2}\beta_i\beta_j K_i K_j W_{E2}(\beta_i,\beta_j,Z_i,Z_j) - K_i K_j W_0(\beta_i,\beta_j,Z_i,Z_j) - 2ru_{io}(1)u_{jo}(1)] \quad (83)$$

when u_{io} and u_{jo} are odd where $u_{io}(1)$ and $u_{jo}(1)$ are given by Eq. (74) and W_{E2} is an integral which is defined in the Appendix.

After the quantities $E_j^{(n)}$ are obtained, the even and odd bending modes can be computed by substitution of Eqs. (70) and (74), respectively, into Eq. (77), and this yields

$$u_n(s) = \sum_{j=1,3,\ldots}^{N'} E_j^{(n)} C_j(\cos\beta_j s - Y_j \cosh\beta_j s)$$
$$(n = 1,3,\ldots,N') \text{ for the even modes} \quad (84)$$

and

$$u_n(s) = \sum_{j=2,4,\ldots}^{N''} E_j^{(n)} K_j(\sin\beta_j s + Z_j \sinh\beta_j s)$$
$$(n = 2,4,\ldots,N'') \text{ for the odd modes} \quad (85)$$

where N' and N'' differ by one, and one of these integers is equal to N.

Discussion of Numerical Results

The first five elastic modes for a particular example are shown in Fig. 5. In the computation for the rotating modes as well as in the computation for the frequency variations which are discussed below, an expansion consisting of a linear combination of the first twenty nonrotating modes was used. The influence of rotation on the mode shapes may be seen by comparing with the nonrotating modes which are also shown in the figures. Table 1 shows the influence of rotation on the frequency parameters for the beam of the above example, and Table 2 shows similar results for larger values of the mass and inertia ratios, r and μ. All of the frequencies are seen to increase with angular velocity. Increasing the tip-mass and rotatory-inertia ratios, r and μ, decreased the frequencies for $\epsilon = 0$; however for the higher rotational speeds this effect is reversed for the third and higher modes. This result is probably attributable to the stiffening influence of the centrifugal forces which overwhelms the influence of the increased tip inertia.

Appendix: Integrals of Transcendental Functions

The following integrals appear in the numerical work. Analytical evaluation of these integrals which are suitable for numerical computation are presented in Appendix C of Ref. 7.

$$W_E(\beta_j,\beta_n,H_j,H_n) \equiv \int_{-1}^{1} x_j x_n ds, \quad (A1)$$

$$W_{E2}(\beta_j,\beta_n,H_j,H_n) \equiv \int_{-1}^{1} s^2 x_j x_n ds \quad (A2)$$

where

$$x_j \equiv \cos\beta_j s + H_j \cosh\beta_j s \quad (A3)$$

Also,

$$W_0(\beta_j,\beta_n,H_j,H_n) \equiv \int_{-1}^{1} y_j y_n ds \quad (A4)$$

$$W_{02}(\beta_j,\beta_n,H_j,H_n) \equiv \int_{-1}^{1} s^2 y_j y_n ds \quad (A5)$$

where

$$y_j \equiv \sin\beta_j s + H_j \sinh\beta_j s \quad (A6)$$

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